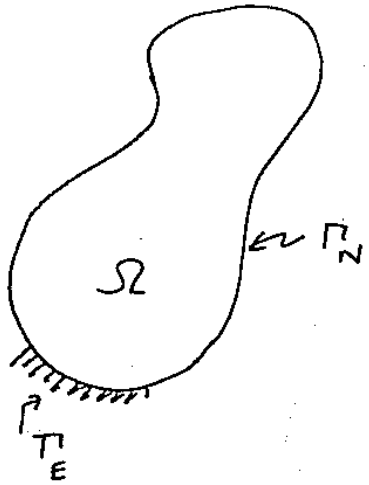
Lecture No. 5



$$L(u) - p(x) = 0 \text{ in } \Omega$$

$$S_E(u) = \overline{g}_E \text{ on } \Gamma_E$$

$$S_N(u) = \overline{g}_N \text{ on } \Gamma_N$$

- For all weighted residual methods

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i$$

- For all (Bubnov) Galerkin methods

$$w_i = \phi_i \quad \text{(test = trial)}$$

<u>Summary of Conventional Galerkin Method</u>

$$S_E(u_B) = \overline{g}_E \text{ and } S_E(\phi_i) = 0, \qquad i = 1, N \text{ on } \Gamma_E$$

$$S_N(u_B) = \overline{g}_N \text{ and } S_N(\phi_i) = 0, \qquad i = 1, N \text{ on } \Gamma_N$$

$$\mathcal{E}_I = L(u_{app}) - p(x)$$

$$< \mathcal{E}_I, w_j > = 0, \qquad j = 1, N$$

Fundamental Weak Weighted Residual Galerkin

- Only satisfy the essential b.c.'s. This makes things a lot easier!

$$S_E(u_B) = \overline{g}_E \text{ and } S_E(\phi_i) = 0 \qquad i = 1, N \text{ on } \Gamma_E$$

- Only require a limited degree of functional continuity. Therefore we can piece together functions to make up ϕ_i. Again this makes the problem much easier.
- Since we're not satisfying the natural b.c.'s, we also consider the error associated with the locations where the natural b.c.'s is violated:

$$\mathcal{E}_I = L(u_{app}) - p(x)$$

$$\mathcal{E}_{B,N} = -S_N(u_{app}) + \overline{g}_N$$

$$< \mathcal{E}_I, w_j > + < \mathcal{E}_{B,N}, w_j > = 0$$

This establishes the fundamental weak form. Thus we have relaxed the admissibility conditions such that only "essential" b.c.'s must be satisfied.

- The Interior and Boundary Error (for the natural boundary) must go to zero by requiring w_j to the orthogonal to these errors. Hence we are no longer "clamping" the natural b.c. down at the time we set up our sequence but we are "driving" the natural b.c. error to zero as the number of trial functions increases.

- Expanding out the fundamental weak form by substituting $\mathcal{E}_I, \mathcal{E}_{B,N}$ and u_{app}.

$$< L(u_B) - p, w_j > + \sum_{j=1}^{N} \alpha_j < L(\phi_j), w_i >_{\Omega} + < \overline{g}_N - S_N(u_B), w_i >_{\Gamma_N}$$

$$- \sum_{j=1}^{N} \alpha_j < S_N(\phi_j), w_j >_{\Gamma_N} = 0 \qquad i = 1, N$$

Symmetrical Weak Weighted Residual Form

- Let's continue to relax admissibility by integrating the fundamental weak form by parts. The "halfway point" of the integration represents the symmetrical weak form".
- This integration by parts procedure:
  - Reduces the required functional continuity on the approximating functions. It reduces functional continuity requirements on the trial functions while increasing them on the test functions. At the symmetrical weak form these requirements are balanced. Thus this is the optimal weak form.
  - Furthermore the unknown functions evaluated at the boundary disappear.
- In general the natural b.c. is only satisfied in an average sense of $N \to \infty$.
- When the operator is self-adjoint, the matrix generated will still be symmetrical for the symmetrical and fundamental weak forms.

Definition and Use of Localized Functions

- Split the domain up into segments over which trial/test functions are defined.

  Motivation, it's easier to satisfy b.c.'s for complex 2D/3D geometries since these can be satisfied locally (for weak formulations we need only satisfy the essential b.c.'s).
- For the finite element method (FEM), we use functions defined over small subdomains, i.e. localized functions.
- However we must ***ensure*** that we satisfy the correct degree of functional continuity. For example, consider:

$$L(u) = \frac{d^2u}{dx^2}$$

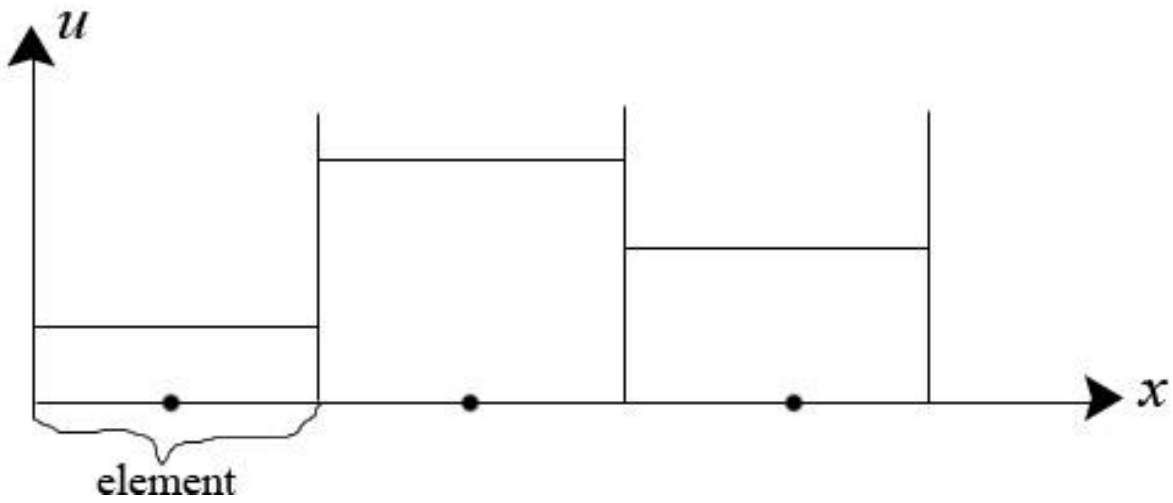
  $W^{(2)}$ space is required for fundamental weak form (continuity of the function and the first derivative and a finite second derivative).

  $W^{(1)}$ space is required for the symmetrical weak form (continuity of the function and a finite first derivative).
- Split the domain up into segments and define trial functions within each element.

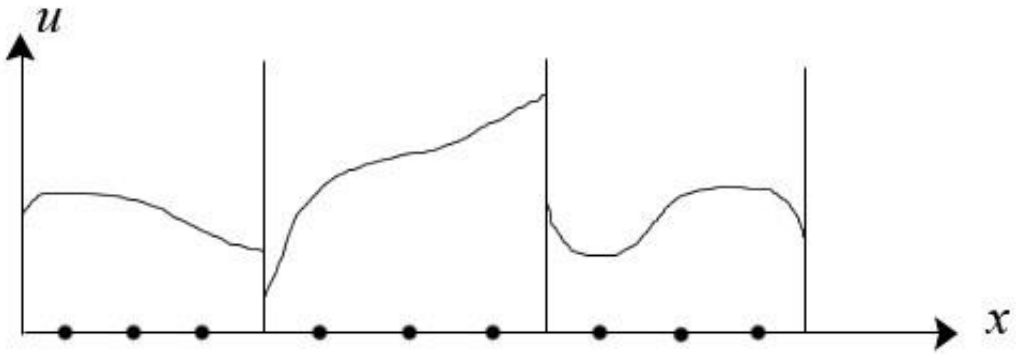
Option 1

No functional continuity but finite function $\rightarrow W^{(0)}$ space

(a) The simplest approximation is a histogram (i.e. constant value of the function over the element). This defines only one unknown per "finite element".



(b) We can define more complex functions over the element leading to more unknown per element (to represent an increased order of polynomial or function within the element). However we still have no functional continuity between the elements and thus these functions still are from the $W^{(0)}$ space.

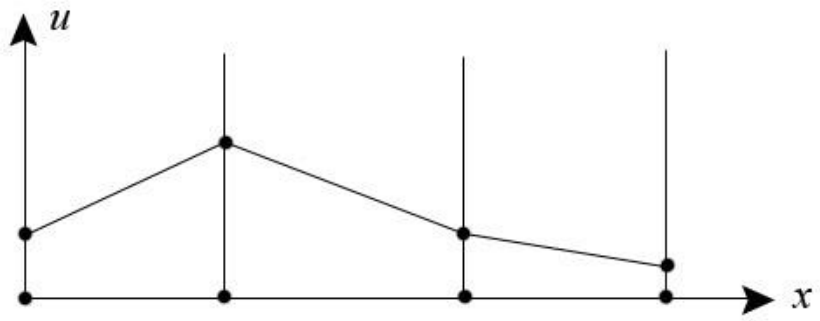


Option 2

The function is continuous which defines the $W^{(1)} = C_o$ space. This space requires no continuity of the derivative. We define the discrete variables equal to the functional values at the nodes. Nodes must always join up at the inter element boundaries!

(a) The simplest approximation consists of a linear approximation over an elements which involves 2 unknowns per element and 1 unknown per node: Thus over each element:

$$u = c_o + c_1 x$$

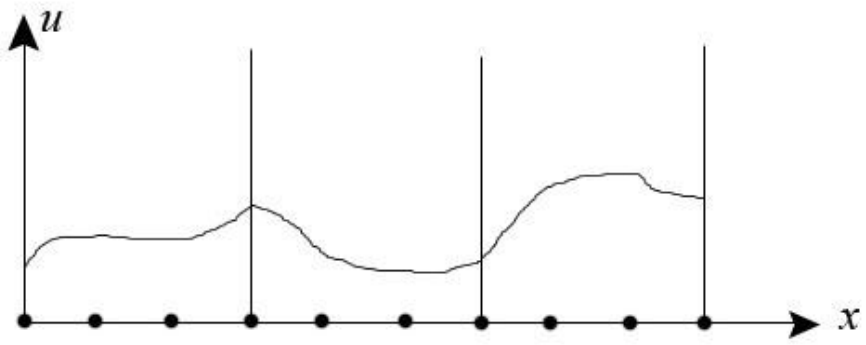


Thus we've gone from an element measure to a nodal measure.

An interpolating function defined with nodal functional values and $W^{(1)} = C_o$ continuity is called a Lagrange Interpolating function.

(b) We can use higher degree functions within each element (i.e. more nodes per element) however functional continuity will still only be $W^{(1)}$. For example:

$$u = c_o + c_1 x + c_2 x^2 + c_3 x^3$$

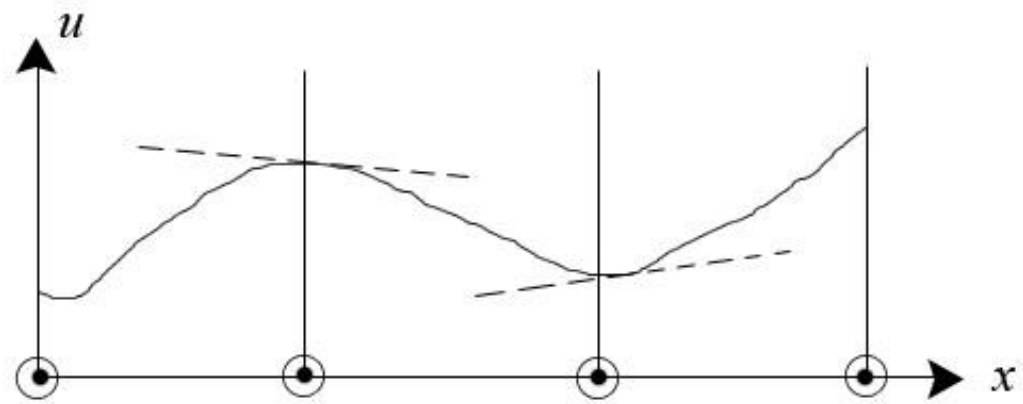


Option 3

The function and first derivative continuous defines the $W^{(2)} = C_1$space. In this case we must define the function and the first derivative as nodal variables.

(a) Simplest approximation involves 2 nodes with 2 unknowns per node. Thus over each element:

$$u = c_o + c_1 x + c_2 x^2 + c_3 x^3$$



Therefore u and $u_{,x}$ are the unknowns. This involves 4 unknowns per element, 2 unknowns per node.

Interpolating functions defined with nodal functional and first derivative values with $W^{(2)} = C_1$ continuity are called Hermite Interpolating functions.

1st order Hermite: u and $u_{,x}$ continuous

2nd order Hermite: u, $u_{,x}$ and $u_{,xx}$ continuous → 3 unknowns per node

Note 1

For the Galerkin method, $w_j = \phi_j$

- The fundamental weak form requires different order functional continuity for ϕ_j as trial functions than the w_j (or ϕ_j as test) functions.
- The symmetrical weak form is obtained by integration by parts until the space requirements for ϕ_j and w_j are the same. ***This is the most attractive form to use***. We note that the number of unknowns has been minimized.

Example

- Fundamental weak form:

$$\int_0^1 \left\{\frac{d^2u_{app}}{dx^2} + u_{app} - x\right\} w_j dx + \left[\left(g - \frac{du_{app}}{dx}\right) w_j\right]_{x=1} = 0$$

$$w_j \in L_2 \text{ and } u_{app} \text{ (or } \phi_i) \in W^{(2)}, \quad j = 1, N$$

Therefore we must choose w_j and $\phi_j \in W^{(2)}$

The functional and first derivative must be continuous) leading to a cubic approximation over the element with 2 *unknowns* per node (function value and its 1st derivative). Thus we require Hermite interpolation

- 2nd weak form: Symmetrical Weak Form

$$\int_0^1 \left(-\frac{du_{app}}{dx}\frac{dw_j}{dx} + uw_j + xw_j\right) dx + \left|gw_j\right|_{x=1} = 0$$

- The opposite of the fundamental weak form is obtained by interchanging u_{app} and w_j and leads to the B.E.M form

$$\left|\frac{dw_j}{dx}u_{app}\right|_0^1 + \left|gw_j\right|_{x=1} + \int_0^1 \left\{uw_j + xw_j + u_{app}\frac{d^2w_j}{dx^2}\right\}dx = 0$$

Now $w_j \in W^{(2)}$ and u_{app} (or ϕ_j) $\in L_2$

Note 2

Alternative strategy for deriving the symmetrical weak form:

- Start with standard Galerkin formulation
- Integrate by parts
- Substitute in for the specified natural b.c.

Example

- Standard Galerkin formulation

$$\int_0^1 \left\{ \frac{d^2 u_{app}}{dx^2} + u_{app} + x \right\} w_j dx = 0$$

- Integrate by parts:

$$\int_0^1 \left\{ -\frac{du_{app}}{dx}\frac{dw_j}{dx} + uw_j + xw_j \right\} dx + \left[\frac{du_{app}}{dx} w_j \right]_{x=0}^{x=1} = 0$$

$$\Rightarrow$$

$$\int_0^1 \left\{ -\frac{du_{app}}{dx}\frac{dw_j}{dx} + uw_j + xw_j \right\} dx + \left[\frac{du_{app}}{dx} w_j \right]_{x=1} - \left[\frac{du_{app}}{dx} \right]_{x=0} = 0$$

- However for our example case:

$$\left.\frac{du_{app}}{dx}\right|_{x=1} \cong g \qquad \text{(specified natural b.c.)}$$

- Substituting the above equation as well as $w_j\big|_{x=0} = 0$ (since ϕ_j satisfies the homogeneous form of the essential b.c.), leads to:

$$\int_0^1 \left\{\frac{du_{app}}{dx}\frac{dw_j}{dx} + uw_j + xw_j\right\} dx + \left[gw_j\right]_{x=1} = 0$$

Note 3

For collocation we cannot use an integration by parts procedure to lower the required degree of functional continuity as we could for Galerkin methods. This is due to the form of the weighting functions, $w_j = \delta(x - x_j)$ (the dirac delta function) which is not differentiable.

$$\frac{d\delta(x - x_j)}{dx}$$

does not exist. Therefore it is not possible to find any weak forms!

Therefore for a 2nd order operator we must use Hermite type interpolation compared to the Galerkin Symmetrical weak form for which we only needed Lagrange type interpolation.

Note 4

- Summary of Advantages of Weak Forms:
  - Fundamental weak form has relaxed b.c. requirements. It's much easier to find trial/test functions which satisfy the homogeneous form of the essential b.c. than both the essential and natural b.c.

    This is especially true when using the FE method where we simply go into the matrix and set the essential b.c. u and we need not worry about "strictly" enforcing the natural boundary condition du/dx
  - Symmetrical weak form has relaxed functional continuity requirements. This is very important when defining functions over split domains (FE method).

    Note that one important reason we split the domains is in order to satisfy the essential b.c.'s easier. The symmetrical weak form lowers the inter-element functional continuity requirements and thus lowers the number of unknowns. In addition part of the natural b.c. error falls out due to the way in which it was defined.

Notes on a 4th derivative operator problem

Consider the equilibrium equation for a beam on an elastic foundation

$$EI\frac{d^4v}{dx^4} + kv = p(x)$$

where

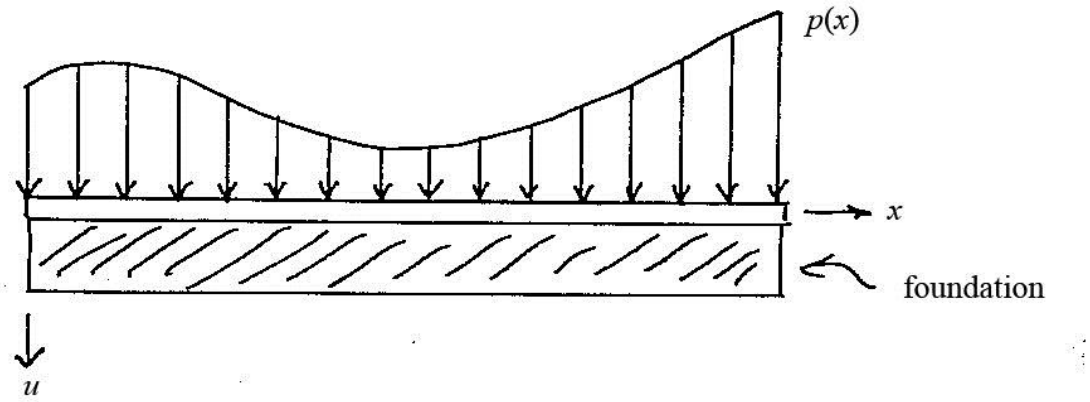
v = beam displacement

E = modulus of elasticity

I = moment of inertia

k = foundation constant

p = distributed load on the beam



Standard Galerkin

$$\langle L(v) - p, \delta v \rangle = 0$$

$$\Rightarrow$$

$$\int_0^l \left\{ EI \frac{d^4 v}{dx^4} + kv - p \right\} \delta v dx = 0$$

- We can establish what the essential and natural b.c.'s are by using the integration by parts procedure to find:

$$\langle L(v), \delta v \rangle = \langle L^*(\delta v), v \rangle +$$

$$\int [F_1(\delta v) G_1(v) + F_2(\delta v) G_2(v) - F_1(v) G^*{}_1(\delta v) - F_2(v) G^*{}_2(\delta v)] d\Gamma$$

- We perform 2 integration by parts to find the bc's (assuming EI constant):

$$\langle L(v), \delta v \rangle = \int_0^l \left\{ EI \frac{d^2 v}{dx^2} \frac{d^2 \delta v}{dx^2} + kv\delta v \right\} dx + \left[-EI \frac{d^2 v}{dx^2} \frac{d\delta v}{dx} + EI \frac{d^3 v}{dx^3} \delta v \right]_{x=0}^{x=1}$$

- This is halfway point of the integration by parts procedure and represent the symmetrical weak form:
- Hence the essential b.c.' are: $F_1(v) = v \rightarrow$ displacement

$$F_2(v) = \frac{dv}{dx} \rightarrow \text{ slope}$$

- the natural b.c.'s are: $G_1(v) = -EI\frac{d^3v}{dx^3} = \overline{Q} \rightarrow$ applied shear

$$G_2(v) = +EI\frac{d^2v}{dx^2} = \overline{M} \rightarrow \text{applied moment}$$

- ***Note that the weighting functions that are to be used in establishing the fundamental weak form <u>fall out of this integration by parts procedure</u> used in establishing self-adjointness of the operator (which also establishes b.c.'s). You want terms to drop later on!***

- Natural boundary errors
  - The error in the moment is:

$$\mathcal{E}_B' = \left\{ EI \frac{d^2 v}{dx^2} - \overline{M} \right\} \left\{ \frac{d\delta v}{dx} \right\}$$

  - A error in the shear is:

$$\mathcal{E}_B'' = \left\{ -EI \frac{d^3 v}{dx^3} - \overline{Q} \right\} \{ \delta v \}$$

- Weighted natural boundary errors
  - A moment error is weighted by a rotation:

$$\langle \mathcal{E}_B', \frac{d\delta v}{dx} \rangle = \int_{\Gamma_N} \left\{ EI \frac{d^2 v}{dx^2} - \overline{M} \right\} \left\{ \frac{d\delta v}{dx} \right\} d\Gamma$$

  - A shear error is weighted by a displacement

$$\langle \mathcal{E}_B'', \delta v \rangle = \int_{\Gamma_N} \left\{ -EI \frac{d^3 v}{dx^3} - \overline{Q} \right\} \{ \delta v \} \, d\Gamma$$

- Note that we must specify essential b.c.'s somewhere!
- Boundary conditions must be specified in pairs for this problem
  - Displacement and rotation are paired
  - Shear and moment are paired
- The error/weighting products fall out of the integration by parts process. They are also dimensionally consistent with all terms in the total error equation.
- The total error equation for this problem is the fundamental weak form

$$\langle \mathcal{E}_I^{\square}, \delta v \rangle + \langle \mathcal{E}_B', \frac{d\delta v}{dx} \rangle + \langle \mathcal{E}_B'', \delta v \rangle = 0$$

where

$$\mathcal{E}_I^{\square} = EI \frac{d^4 v}{dx^4} + kv - p$$

- The symmetrical weak form is established by integrating the fundamental weak form by parts.